SOME PROBLEMS ON BASES IN BANACH AND FRECHET SPACES*

BY

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ABSTRACT

This paper is a revised version of the author's report during the Symposium "On Series and Geometry in Linear Spaces" held at The Hebrew University of Jerusalem, March 16–24, 1964. Some problems of the existence (for a given Frechet space) of closed linear subspaces and quotient spaces with bases are discussed.

In this report we present some results of the forthcoming papers of M. I. Kadec and the author [17] and I. Singer and the author [21] and discuss some open questions concerning bases in Banach and Frechet space.

1. **Subspaces and over-spaces with bases.** Let X be a separable infinite-dimensional Fréchet space. Then there exist infinite-dimensional Fréchet spaces X_0 and X_1 with bases and embedding isomorphisms i_0 and i_1 such that the diagram

$$X_1 \overset{i_1}{\to} X \overset{i_0}{\to} X_0 \tag{1}$$

holds.

We can put $X_0 = C$ for X being a Banach space and $X_0 = (C \times C \times \cdots)_s$ for arbitrary separable Fréchet space(¹): The existence of X_1 (for Banach spaces) is mentioned in the monograph of Banach [2, p. 238]. There are several proofs of this result: Bessaga and Pełczyński [4], [5], Gelbaum [13], Day [9]. The proof of Day gives something more. It is shown that in every infinite-dimensional Banach space X there is a biorthogonal system (x_n^*, x_n) such that

$$\| x_n^* \| = \| x_n \| = 1 \qquad (n = 1, 2, \cdots)$$

and (x_n) is a basic sequence ,i.e. is a basis for a subspace of X. The proofs of

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(¹) By Fréchet space we mean a linear-metric, complete, locally-convex space. The symbol C denotes the Banach space of all real valued continuous functions on the closed interval [1;0] s denotes the Fréchet space of all real valued sequences: $(C \times C \times \ldots)_s$ denotes the space of all sequences with elements in C.

Bessaga and Pełczyński are valid for Fréchet spaces. If X is a Fréchet space which is not isomorphic to any Banach space, then it is possible to choose X_1 nuclear [8]. We do not know how "nice" one can choose X_0 for a given X.

P.1. Let X be a separable Fréchet space (in particular a Banach space) having one the of properties (A_i) $(i = 1,2,3,4,5,6)$. Is X isomorphically embeddable in a Fréchet space (respectively in a Banach space) with a basis and having the same property (A_i)? The properties considered are:

(A_1) X is reflexive;

(A_2) X is weakly complete;

(A_3) X is isomorphic to a conjugate space;

(A_4) bounded sets (2) in X are weakly conditionally sequentially compact;

(A_5) bounded sets in X are precompact ($= X$ is a Montel space);

(A_6) X is nuclear.

In an arbitrary Fréchet space there are many different basic sequences. Recent results of M. I. Kadec and Pełczyński [17] give rather satisfactory information in what cases in a given set of vectors there exist basic sequences.

Definition. Let X be a Fréchet space and let X^* be the conjugate space to X. A set $\Gamma \subset X^*$ is said to be norming for X if for every fundamental system of bounded sets (B_n) in X^* the sequence $(\|\cdot\|_{B_n \cap \Gamma})$, with $\|x\|_{B_n \cap \Gamma} = \sup\{|x^*x| : x^* \in B_n \cap \Gamma\}$ for $x \in X$ is an admissible system of pseudonorms in X, i.e. consistent with the topology of X.

We shall write $x \underset{\Gamma}{\to} 0$ iff $x^*x_n \to 0$ for every x^* in Γ.

THEOREM 1. *A subset M in an arbitrary Fréchet space X contains an infinite basic sequence if and only if there is a norming set $\Gamma \subset X^*$ for X and a sequence (x_n) of elements of M such that $x_n \neq 0$ $(n = 1, 2, \cdots)$ and $t_n x_n \underset{\Gamma}{\to} 0$ for every sequence of scalars (t_n).*

THEOREM 2. *Let (x_n) be a sequence in a Fréchet space X which does not converge to 0 and let $x_n \underset{\Gamma}{\to} 0$ for some subset $\Gamma c X^*$ which is norming for X. Then (x_n) contains an infinite basic sequence.*

THEOREM 3. *Let M be a bounded non compact set in a Fréchet space X. Then there exists in X a subspace with basis, which is spanned by an infinite part of M.*

THEOREM 4. *A Fréchet space X has one of the properties A_i for $i = 1,2,4,5$, if and only if every subspace of X with a basis has the same property A_i (cf. [20]).*

THEOREM 5. *Let M be a bounded set in a Fréchet space X. Then the following conditions are equivalent:*

(2) We recall that a set B in a topological space X is said to be bounded if the function $f: [1; 1] \times B \to X$, with $f(t, x) = t\,x$, is uniformly continuous.

(1) *The weak closure of M is not weakly compact in X.*

(2) *There is a closed linear subspace E of X with a basis, such that the weak closure of* $E \cap M$ *is not weakly compact.*

(3) *There is a basic sequence* (x_n) *in M and a linear functional* x_0 *in* X^* *such that* $\liminf_n x_0^* x_n > 0$.

Some results of R. C. James (Theorems 1–3 in [22]) can be easily deduced from our Theorem 5 (cf. [25]).

We note (compare with Theorem 3) that if it were true that every subset M of a Fréchet space X contains an infinite part which spans a subspace with a basis, then in every separable Fréchet space there would exist a basis. This follows from the fact that in every separable Fréchet space there is a sequence of elements such that each infinite subsequence of it spans the whole space X [23]. This remark is due to V. I. Gurarij.

P.2a. Is a Fréchet space nuclear if and only if each of its subspaces with a basis is nuclear?

P.2b. Is a Banach space isomorphic to a Hilbert space if and only if each of its subspaces with a basis is isomorphic to a Hilbert space?

The next two problems concern conjugate spaces (the property A_3)

P.3. Does every infinite-dimensional conjugate Banach space contain an infinite-dimensional subspace isomorphic to a conjugate space and having a basis (in particular, a boundedly complete basis)?

P.4. Does every separable, non quasi-reflexive, conjugate space contain a subspace which is not isomorphic to a conjugate space?

Recently J. Lindenstrauss has constructed in the space l a subspace non isomorphic to a conjugate space [24].

2. The dual problem. It is natural to consider the dual diagram to (1). Namely

P.5. Do there exist, for every infinite-dimensional separable Fréchet space X, infinite-dimensional Fréchet spaces X^0 and X^1 with bases, and linear operators onto h_0 and h_1 such that the diagram

$$X^1 \overset{h_1}{\leftarrow} X \overset{h_0}{\leftarrow} X^0 \tag{1}$$

holds?

The only result known on P.5. is:

If X is a Banach space then we can put $X^0 = l$, and if X is a Fréchet space not isomorphic to a Banach space then we can put $X^1 = s$ (this uses a result of Eidelheit [12]). For every reflexive space X a space X^1 exists (using standard dual arguments and applying diagram (1) for conjugate spaces). Thus a space X^1, and h_1, exist for every Banach space which admits a linear mapping onto a reflexive space (this property is equivalent to the fact that X^* contains a reflexive subspace).

P.6. Is it true that every reflexive space is a linear image of a reflexive space with a basis?

For Banach spaces P.6 is equivalent to the problem P.1 for the property A_1.

The next problem is connected with P.1. for A_6.

P.7. Is it true that every nuclear space is a linear image of a nuclear space with a basis?

3. **Conditional and unconditional basic sequences.** We recall that a basic sequence (x_n) in X is called unconditional if it is an unconditional basis for a subspace of X. A non unconditional basic sequence is called conditional. An important and rather difficult question is (cf. [6], [9])

P.8. Does there exist in every infinite dimensional Banach space X an infinite dimensional subspace with an unconditional basis?

If X is a Fréchet space which is not isomorphic to a Banach space then X contains an infinite-dimensional subspace with an unconditional basis (either (s) or a Kothe space, cf. [8, Theorem 1]). The answer on P.8. is positive in the case where X is isomorphically embeddable into a Banach space with an unconditional basis [5, p. 157]. It is perhaps easier to find a (positive) solution of P.8. under the additional assumption that X is uniformly convex or that X admits a linear mapping onto a space with an unconditional basis. The problem P.8. is important from the point of view of the topological classification of separable Banach spaces (cf. [7]). For this purpose it is sufficient to answer affirmatively the following

P.9. Does every hereditary non reflexive Banach space contain a subspace either isomorphic to l or to c_0?

From a result of R.C. James [14] it follows that P.9. is a particular case of P.8.

Altough we do not know whether there exist unconditional basic sequences in every Fréchet space, the construction of basic sequences about which we can establish that the are conditional is also difficult. To show that in every infinite-dimensional Banach space there exist conditional basic sequences V. I. Gurarij [15] used the deep result of Dvoretzky [10] on the spherical sections of convex bodies and the heavily analytic construction of Babienko [1] of a conditional basis in l_2. Recently I. Singer and the author [21] have obtained a considerably stronger result than that of Gurarij.

THEOREM 6. *In every infinite-dimensional Banach space with a basis there exist continuum non equivalent normalized bases from which at least one is conditional.*

The proof of Theorem 6 given in [21], though not simple, does not make use of the result of Dvoretzky [10].

We recall that bases (x_n) and (y_n) are said to be equivalent if $\sum t_n x_n$ converges if and only if $\sum t_n y_n$ converges for every sequence of scalars (t_n). If $\| x_n \| = 1$ $(n = 1, 2, \cdots)$ then a basis (x_n) is called normalized.

By a theorem of Dynin and Mitjagin [11] (see also [18]) every basic sequence in a nuclear Fréchet space is unconditional. It seems to be very probable that the results of Gurararij and Pełczyński-Singer quoted above can be generalized to a new characterization of nuclearity of Fréchet spaces. Namely

P.10. Let X be a Fréchet space (with a basis). Is it true that X is nuclear if and only if every basic sequence (every basis) in X is unconditional?

It is well known that in l_2 all normalized unconditional bases are equivalent (Bari [3], Gelfand [14]).

P.11. Is it true that an infinite-dimensional Banach space X with an uncon ditional basis having the property that all normalized unconditional bases in X are equivalent is isomorphic to l_2?

However we are not able at present to construct any normalized unconditional basis in the spaces l and c_0 non equivalent with that consisting of unit vectors. Such bases for the spaces l_p $(1 < p < +\infty)$ were constructed in [19].

Let us mention

Theorem 7 (Pełczyński-Singer [21]). *If X is a Banach space with an unconditional basis and all normalized infinite unconditional basic sequences in X are equivalent, then X is either isomorphic to l_2 or is of finite dimension.*

4. Hilbertian and Besselian bases and basic sequences.

Definiton. A normalized basis (x_n) (basic sequence) in a Banach space X is called Hilbertian (Besselian) provided that $\sum_n t_n x_n$ converges for every $(t_n) \in l_2$ (if $\sum t_n x_n$ converges, then $(t_n) \in l_2$).

It is rather easy, using the method of Gurarij [15], to show that every Banach space in which all normalized basic sequences are Hilbertian (Besselian) is of finite dimension. We are not able to establish the result analogous to Theorem 6. Hence

P.12. Let X be a Banach space with a basis and let all normalized bases in X be Hilbertian (Basselian). Is it then true that X is of a finite dimension?

Let us mention

P.13. Does there exist in the space C (in the space L) a normalized Besselian (resp. Hilbertian) basis? In particular, does there exist in C an orthonormal system of continuous and uniformly bounded functions which is a basis for C?

We conjecture that the answer on P.13 is "no".

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